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TORSION-FREE COVERS II

BY

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ABSTRACT

This paper continues the study of the existence of torsion-free covers with respect to a faithful hereditary torsion theory $(\mathfrak{T}, \mathfrak{F})$ of left modules over a ring R with unity. If the filter of left ideals associated with $(\mathfrak{T}, \mathfrak{F})$ has a cofinal subset of finitely generated left ideals, then every left R-module has a torsion-free cover. An example is given to illustrate how this result generalizes all previously known existence theorems for torsion-free covers.

In this note R denotes a ring with unity, and all modules are unital left R-modules.

Let $(\mathfrak{T}, \mathfrak{F})$ be a hereditary torsion theory of R-modules. (See [5, 6, 7, 8] for definitions and properties.) The class $\mathfrak{F}$ of torsionfree modules is closed under taking submodules, injective envelopes, and direct sums. $(\mathfrak{T}, \mathfrak{F})$ is called faithful if $R \in \mathfrak{F}$. Associated with $(\mathfrak{T}, \mathfrak{F})$ is a topologizing and idempotent filter $F(\mathfrak{T}) = \{I \mid R/I \in \mathfrak{T}\}$ of left ideals of R. An epimorphism $\Psi: M' \to M$ is called an $\mathfrak{F}$-precover of M if it has the following properties:

(i) $M' \in \mathfrak{F}$;

(ii) for each homomorphism $\phi: F \to M$ with $F \in \mathfrak{F}$, there exists a homomorphism $f: F \to M'$ such that $\Psi \circ f = \phi$.

If $F \in \mathfrak{F}$, a submodule N of F is said to be pure in F if and only if $F/N \in \mathfrak{F}$. An $\mathfrak{F}$-precover $\Psi: M' \to M$ is called an $\mathfrak{F}$-cover of M if ker Ψ contains no nonzero pure submodules of M'. If every module has an $\mathfrak{F}$-cover, $(\mathfrak{T}, \mathfrak{F})$ is said to be universally covering.

$\mathfrak{F}$-covers were first defined by Enochs [3] and shown to exist for the usual torsion theory over an integral domain. These results were extended to perfect torsion theories by Banaschewski [1]. Teply [8] showed that faithful hereditary torsion theories $(\mathfrak{T}, \mathfrak{F})$, for which the direct sum of injective modules in $\mathfrak{F}$ is injective, are universally covering. The existence and properties of covers is

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further discussed by Enochs [4] and by Golan and Teply [6]. Finally, Cheatham [2] characterizes the left nonsingular rings for which the singular (Goldie) torsion theory is universally covering.

The main theorem of this note generalizes all of the above results. An elementary example is provided to illustrate this generalization.

Before stating the main theorem, we need one preliminary result.

LEMMA. *Let* $(\mathfrak{I}, \mathfrak{F})$ *be a faithful hereditary torsion theory such that* $F(\mathfrak{I})$ *has a cofinal subset of finitely generated left ideals, and let* M *be an injective module.* $\Psi: M' \to M$ *is an* $\mathfrak{F}$*-precover if and only if* $M' \in \mathfrak{F}$ *and, for each injective* $E \in \mathfrak{F}$ *and each* $\phi'': E \to M$ *such that* $\ker \phi''$ *contains no nonzero pure submodules of* E, *there exists* $f': E \to M'$ *such that* $\Psi \circ f' = \phi''$.

PROOF. The "only if" part is trivial; so we prove the "if" part. Suppose that $F \in \mathfrak{F}$ and $\phi: F \to M$. By [8, lemma 2.1 (3)], the union of pure submodules of F in $\ker \phi$ is pure in F. So by Zorn's lemma, there exists a pure submodule P maximal among the pure submodules of F contained in $\ker \phi$. Hence there exists $\phi': F/P \to M$ such that $\phi = \phi' \circ \eta$, where $\eta: F \to F/P$ is the natural homomorphism. By [8, prop. 2.1 (2)], $\ker \phi'$ contains no nonzero pure submodules of F/P.

Since M is injective ϕ' extends to $\phi'': E(F') \to M$, where $F' = F/P$. If $\ker \phi''$ contains a pure submodule H of $E(F')$, then $(F'+H)/H \subseteq E(F')/H \in \mathfrak{F}$; so $F'/(F' \cap H) \in \mathfrak{F}$ and $F' \cap H \subseteq \ker \phi'' \cap F' = \ker \phi'$. Hence $F' \cap H = 0$. Since F' is essential in $E(F')$, $H = 0$. Now by the hypothesis, there exists $f': E(F') \to M'$ such that $\Psi \circ f' = \phi''$. Hence $\phi = \phi' \circ \eta = \phi'' \circ \eta = \Psi \circ f' \circ \eta$. By setting $f = f' \circ \eta$, we see that $\Psi: M' \to M$ is an $\mathfrak{F}$-precover.

NOTATION. If $f: M \to N$ is a homomorphism and if $X \subseteq M$, then $f \mid X$ denotes the restriction of f to X.

THEOREM. *If* $(\mathfrak{I}, \mathfrak{F})$ *is a faithful hereditary torsion theory such that* $F(\mathfrak{I})$ *has a cofinal subset of finitely generated left ideals, then* $(\mathfrak{I}, \mathfrak{F})$ *is universally covering.*

PROOF. To show that every module has an $\mathfrak{F}$-precover, it is sufficient to show that each injective module M has an $\mathfrak{F}$-precover by [8, lemma 2.2 (1)]. Thus we assume M is an injective module.

Let $\{E_\alpha\}_{\alpha \in A}$ be a set of representatives of the isomorphism classes of injective hulls of cyclic modules in $\mathfrak{F}$. For each $\alpha \in A$ and each $g \in \mathrm{Hom}(E_\alpha, M)$, let $E_{\alpha g}$ be a copy of E_α. Define $X = \bigoplus_{\alpha \in A} (\bigoplus_{g \in \mathrm{Hom}(E_\alpha, M)} E_{\alpha g})$. For each $k \in$

$\mathrm{Hom}(E(X), M)$, let X_k be a copy of $E(X)$. Let $M' = \bigoplus_{k \in \mathrm{Hom}(E(X), M)} X_k$, and define $\Psi: M' \to M$ via $\Psi \mid X_k = k$ for each k. We wish to show that $\Psi: M' \to M$ is an $\mathfrak{F}$-precover for M.

Since $M' \in \mathfrak{F}$, the lemma implies that $\Psi: M' \to M$ is an $\mathfrak{F}$-precover if, for any nonzero injective $F \in \mathfrak{F}$ and any $\phi: F \to M$ such that $\ker \phi$ contains no nonzero pure submodules of F, there exists $f: F \to M'$ such that $\Psi \circ f = \phi$. Let $\bigoplus_{\alpha \in B} F_\alpha$ be a direct sum of injective hulls of nonzero cyclic modules such that $\bigoplus_{\alpha \in B} F_\alpha$ is essential in F. Define $F_\alpha \sim F_\beta$ $(\alpha, \beta \in B)$ if and only if there exists an isomorphism $\pi_{\alpha\beta}: F_\alpha \to F_\beta$ such that $(\phi \mid F_\alpha) = (\phi \mid F_\beta) \circ \pi_{\alpha\beta}$. It is easy to check that $\sim$ is an equivalence relation on $\{F_\alpha\}_{\alpha \in B}$.

If $\alpha \neq \beta$ and $F_\alpha \sim F_\beta$, then define $G_{\alpha\beta} = \{x - \pi_{\alpha\beta}(x) \mid x \in F_\alpha\}$. It is easy to check that $G_{\alpha\beta}$ is a submodule of F and that $\theta: G_{\alpha\beta} \to F_\beta: x - \pi_{\alpha\beta}(x) \to \pi_{\alpha\beta}(x)$ is an isomorphism. But for $x - \pi_{\alpha\beta}(x) \in G_{\alpha\beta}$, $(\phi \mid G_{\alpha\beta})(x - \pi_{\alpha\beta}(x)) = \phi(x) - \phi(\pi_{\alpha\beta}(x)) = 0$ by the definition of $\sim$. Thus $G_{\alpha\beta}$ is an injective (and hence pure) submodule of F contained in $\ker \phi$, which contradicts our assumption. Therefore, each equivalence class of $\sim$ has only one member.

For each $\alpha \in B$, there exists an isomorphism $\theta_{\alpha\beta}: F_\alpha \to E_\beta$ for some $\beta \in A$. Define $f' = \bigoplus \theta_{\alpha\beta}: \bigoplus_{\alpha \in B} F_\alpha \to X$ via $F_\alpha \xrightarrow{\theta_{\alpha\beta}} E_{\beta(\phi\theta_{\alpha\beta}^{-1})}$. f' is a monomorphism since each equivalence class of $\sim$ has only one member F_α in it. Extend f' to $f'': F \to E(X)$ by injectivity. Since $\bigoplus_{\alpha \in B} F_\alpha$ is essential in F, then f'' is a monomorphism. By the injectivity of M, choose a homomorphism $g: E(K) \to M$ such that $g \circ f'' = \phi$. Finally, define $f: F \to M'$ via $f = j \circ f''$, where j is the natural injection of X_g into M'. Then $(\Psi \circ f)(x) = \Psi(f(x)) = \Psi(j(f''(x)) = (g \circ f'')(x) = \phi(x)$ for all $x \in F$; hence $\Psi \circ f = \phi$. Since $R \in \mathfrak{F}$, then Ψ must be an epimorphism. Therefore, $\Psi: M' \to M$ is an $\mathfrak{F}$-precover of M.

The above proof shows that, if $F(\mathfrak{I})$ has a cofinal subset of finitely generated left ideals, then every injective module has an $\mathfrak{F}$-precover. Thus we now drop our assumption that M is injective; i.e. let M be any module, and let $\Psi: M' \to M$ be any $\mathfrak{F}$-precover of M. By [8, prop. 2.1 (3)], the union of a chain of pure submodules of M' contained in $\ker \Psi$ is pure in M'. By Zorn's lemma, there is a maximal pure submodule N among the set of pure submodules of F contained in $\ker \Psi$. By [8, lemma 2.2 (3)], the natural epimorphism $\bar{\Psi}: M'/N \to M$ is an $\mathfrak{F}$-precover of M. If follows from [8, prop. 2.1 (2)] that $\ker \bar{\Psi}$ contains no nonzero pure submodules of M'/N; hence $\bar{\Psi}: M'/N \to M$ is an $\mathcal{F}$-cover of M.

Example. Let A be a ring, and let I be a two-sided ideal of A such that $I \neq A$. Let R be the ring of all matrices of the form

$$\begin{pmatrix} A & A/I \\ 0 & A/I \end{pmatrix}.$$

Let $F(\mathfrak{I})$ consist of all left ideals containing the finitely generated, idempotent, left ideal

$$K = \begin{pmatrix} A & A/I \\ 0 & 0 \end{pmatrix}.$$

Since K is also a right ideal of R, then $F(\mathfrak{I})$ is a filter for a hereditary torsion theory, $\mathfrak{I}$ consists of all modules annihilated by K, and $R \in \mathfrak{I}$. Hence the theorem implies that $(\mathfrak{I}, \mathfrak{F})$ is universally covering.

If A is not left noetherian, then A has an infinite ascending chain $\{I_n\}_{n=1}^{\infty}$ of left ideals; so the left ideals of R of the form

$$J_n = \begin{pmatrix} I_n & A/I \\ 0 & A/I \end{pmatrix}$$

are an infinite ascending chain in R such that $R/J_n \in \mathfrak{F}$ for each n. By [8, theor. 1.2], some direct sum of injective modules in $\mathfrak{F}$ is not injective. Therefore, neither [8, theor. 2.4] nor [6, corol. 3.10] can be applied to show that $(\mathfrak{I}, \mathfrak{F})$ is universally covering.

If A is a left nonsingular ring and I is essential as a left ideal of A, then K is not $\mathfrak{I}$-projective in the sense of [5]; so by [5, prop. 16.3], the localization functor associated with $(\mathfrak{I}, \mathfrak{F})$ is not exact. Therefore, $(\mathfrak{I}, \mathfrak{F})$ is not perfect. (See [5, prop.17.1] or [7, theor. 13.1].) Thus neither the results of [1] nor [6, corol. 3.6] nor [6, theor. 3.13] apply to show that $(\mathfrak{I}, \mathfrak{F})$ is universally covering.

If A is commutative semiprime ring and I is an essential maximal ideal of A, then some routine computation shows that the localization of R with respect to $(\mathfrak{I}, \mathfrak{F})$ is R itself; so [6, theor. 3.4] and [6, corol. 3.12] cannot be used in this situation. Moreover, if A is also non-noetherian, then [6, corol. 3.7] cannot be used to determine if $(\mathfrak{I}, \mathfrak{F})$ is universally covering.

Thus the theorem of this note shows that $(\mathfrak{I}, \mathfrak{F})$ is universally covering, but no previously known result can be used to determine if $(\mathfrak{I}, \mathfrak{F})$ is universally covering in this example.

References

1. B. Banaschewski, *On coverings of modules*, Math. Nachr. **31** (1966), 57–71.
2. T. Cheatham, *Finite dimensional torsion free rings*, Pacific J. Math. **39** (1971), 113–118.
3. E. Enochs, *Torsion free covering modules*, Proc. Amer. Math. Soc. **14** (1963), 884–889.
4. E. Enochs, *Torsionfree covering modules II*, Arch. Math. (Basel) **22** (1971), 37–52.

5. J. S. Golan, *Localization of noncommutative rings*, Comm. Pure Appl. Math. **30** (1975).

6. J. S. Golan and M. L. Teply, *Torsion-free covers*, Israel J. Math. **15** (1973), 237–256.

7. B. Stenström, *Rings and Modules of Quotients*, Lecture Notes in Mathematics 237, Springer-Verlag, New York, 1971.

8. M. L. Teply, *Torsionfree injective modules*, Pacific J. Math. **28** (1969), 441–453.

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